

# Averaging 2-Rainbow Domination and Roman Domination

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## Abstract

For a graph  $G$ , let  $\gamma_{r2}(G)$  and  $\gamma_R(G)$  denote the 2-rainbow domination number and the Roman domination number, respectively. Fujita and Furuya (Difference between 2-rainbow domination and Roman domination in graphs, Discrete Applied Mathematics 161 (2013) 806-812) proved  $\gamma_{r2}(G) + \gamma_R(G) \leq \frac{6}{4}n(G)$  for a connected graph  $G$  of order  $n(G)$  at least 3. Furthermore, they conjectured  $\gamma_{r2}(G) + \gamma_R(G) \leq \frac{4}{3}n(G)$  for a connected graph  $G$  of minimum degree at least 2 that is distinct from  $C_5$ . We characterize all extremal graphs for their inequality and prove their conjecture.

**Keywords:** Rainbow domination; Roman domination

**MSC2010:** 05C69

## 1 Introduction

We consider finite, simple, and undirected graphs and use standard terminology and notation.

Rainbow domination of graphs was introduced in [1]. Here we consider the special case of 2-rainbow domination. A *2-rainbow dominating function* of a graph  $G$  is a function  $f : V(G) \rightarrow 2^{\{1,2\}}$  such that  $\bigcup_{v \in N_G(u)} f(v) = \{1,2\}$  for every vertex  $u$  of  $G$  with  $f(u) = \emptyset$ . The *weight* of  $f$  is  $\sum_{u \in V(G)} |f(u)|$ . The *2-rainbow domination number*  $\gamma_{r2}(G)$  of  $G$  is the minimum weight of a 2-rainbow dominating function of  $G$ . Roman domination was introduced in [5]. A *Roman dominating function* of a graph  $G$  is a function  $g : V(G) \rightarrow \{0, 1, 2\}$  such that every vertex  $u$  of  $G$  with  $g(u) = 0$  has a neighbor  $v$  with  $g(v) = 2$ . The *weight* of  $g$  is  $\sum_{u \in V(G)} g(u)$ . The *Roman domination number*  $\gamma_R(G)$  of  $G$  is the minimum weight of a Roman dominating function of  $G$ .

The definitions of the above two types of dominating functions have some obvious similarities; vertices contribute either 0 or 1 or 2 to the weight of these functions; for vertices that contribute 0,

their neighbors contribute at least 2 in total; vertices that contribute 1 do not impose any condition on their neighbors; and vertices that contribute 2 satisfy the requirements of all their neighbors that contribute 0. Nevertheless, while vertices that contribute 1 are useless for their neighbors in Roman domination, they can satisfy ‘half’ the requirements of their neighbors in 2-rainbow domination.

As observed in [3, 6] the two domination parameters are related by the following simple inequalities

$$\gamma_{r2}(G) \leq \gamma_R(G) \leq \frac{3}{2}\gamma_{r2}(G). \quad (1)$$

In fact, if  $g$  is a Roman dominating function of a graph  $G$  of weight  $w$ , then

$$f : V(G) \rightarrow 2^{\{1,2\}} : u \mapsto \begin{cases} \emptyset & , \text{ if } g(u) = 0, \\ \{1\} & , \text{ if } g(u) = 1, \text{ and} \\ \{1, 2\} & , \text{ if } g(u) = 2. \end{cases}$$

is a 2-rainbow dominating function of  $G$  of weight  $w$ , which implies  $\gamma_{r2}(G) \leq \gamma_R(G)$ . Similarly, if  $f$  is a 2-rainbow dominating function of  $G$  of weight  $w$  such that  $|f^{-1}(\{1\})| \geq |f^{-1}(\{2\})|$ , then

$$g : V(G) \rightarrow \{0, 1, 2\} : u \mapsto \begin{cases} 0 & , \text{ if } f(u) = \emptyset, \\ 1 & , \text{ if } f(u) = \{1\}, \text{ and} \\ 2 & , \text{ if } f(u) \in \{\{2\}, \{1, 2\}\}. \end{cases}$$

is a Roman dominating function of  $G$  of weight at most  $3w/2$ , which implies  $\gamma_R(G) \leq \frac{3}{2}\gamma_{r2}(G)$ .

The following result summarizes known tight bounds for the two parameters [2, 4, 6].

**Theorem 1** *Let  $G$  be a connected graph of order  $n(G)$  at least 3.*

- (i)  $\gamma_{r2}(G) \leq \frac{3}{4}n(G)$  [6].
- (ii)  $\gamma_R(G) \leq \frac{4}{5}n(G)$  [2].
- (iii) *If  $G$  has minimum degree at least 2, then  $\gamma_{r2}(G) \leq \frac{2}{3}n(G)$  [4].*
- (iv) *If  $G$  has order at least 9 and minimum degree at least 2, then  $\gamma_R(G) \leq \frac{8}{11}n(G)$  [2].*

Also bounds on linear combinations of the parameters were considered.

**Theorem 2 (Fujita and Furuya [4])** *If  $G$  is a connected graph of order  $n(G)$  at least 3, then  $\gamma_{r2}(G) + \gamma_R(G) \leq \frac{6}{4}n(G)$ .*

In view of Theorem 1(i) and (ii), one would expect an upper bound on  $(\gamma_{r2}(G) + \gamma_R(G))/(2n(G))$  that is somewhere between  $3/4$  and  $4/5$ . Theorem 2 is slightly surprising as it shows that this upper

bound has the smallest possible value, namely  $3/4$ . In fact, (1) and Theorem 2 imply Theorem 1(i).

Our first result is the following.

**Theorem 3** *If  $G$  is a connected graph of minimum degree at least 2 that is distinct from  $C_5$ , then  $\gamma_{r2}(G) + \gamma_R(G) \leq \frac{4}{3}n(G)$ .*

Theorem 3 confirms a conjecture of Fujita and Furuya (Conjecture 2.11 in [4]). Similarly as for Theorem 2, it is again slightly surprising that the upper bound on  $(\gamma_{r2}(G) + \gamma_R(G))/(2n(G))$  in Theorem 3 has the smallest of the possible values suggested by Theorem 1(iii) and (iv), namely  $2/3$ . Note that (1) and Theorem 3 imply Theorem 1(iii).

Another result concerning linear combinations of the parameters is the following.

**Proposition 4 (Chellali and Rad [3])** *There is no constant  $c$  such that  $2\gamma_{r2}(G) + \gamma_R(G) \leq 2n(G) + c$  for every connected graph  $G$ .*

Chellali and Rad posed the problem (Problem 13 in [3]) to find a sharp upper bound on  $2\gamma_{r2}(G) + \gamma_R(G)$  for connected graphs  $G$  of order at least 3. In fact, (1) and Theorem 2 immediately imply the following.

**Corollary 5** *If  $G$  is a connected graph of order  $n(G)$  at least 3, then  $2\gamma_{r2}(G) + \gamma_R(G) \leq \frac{9}{4}n(G)$ .*

Corollary 5 is sharp and hence solves the problem posed by Chellali and Rad.

As our second result we characterize all extremal graphs for Theorem 2, all of which are also extremal for Corollary 5.

## 2 Results and Proofs

Our proof of Theorem 3 relies on an elegant approach from [2]. We also use the reductions described in Lemma 4.1 in [2]. Unfortunately, the proofs of (b) and (c) of Lemma 4.1 in [2] are not completely correct; the graphs  $G'$  considered in these proofs may have vertices of degree less than 2. We incorporate corrected proofs for these reductions as claims within the proof of Theorem 3.

**Lemma 6** *Let  $G$  be a graph that contains an induced path  $P$  of order 5 whose internal vertices have degree 2. If  $G'$  arises from  $G$  by contracting three edges of  $P$ , then  $n(G') = n(G) - 3$ ,  $\gamma_{r2}(G) \leq \gamma_{r2}(G') + 2$ , and  $\gamma_R(G) \leq \gamma_R(G') + 2$ .*

*Proof:* Let  $P : xuvwy$ , that is,  $G'$  arises from  $G$  by deleting  $u$ ,  $v$ , and  $w$ , and adding the edge  $xy$ . Clearly,  $n(G') = n(G) - 3$ .

Let  $f$  be a 2-rainbow dominating function of  $G'$ . If  $f(x), f(y) \neq \emptyset$  or  $f(x) = f(y) = \emptyset$ , then setting  $f(u) = f(w) = \emptyset$  and  $f(v) = \{1, 2\}$  extends  $f$  to a 2-rainbow dominating function of  $G$ .

Now we assume that  $1 \in f(x)$  and  $f(y) = \emptyset$ . If  $f(x) = \{1\}$ , then setting  $f(u) = \emptyset$ ,  $f(v) = \{2\}$ , and  $f(w) = \{1\}$  extends  $f$  to a 2-rainbow dominating function of  $G$ . Finally, if  $f(x) = \{1, 2\}$ , then setting  $f(u) = f(v) = \emptyset$  and  $f(w) = \{1, 2\}$  extends  $f$  to a 2-rainbow dominating function of  $G$ . By symmetry, this implies  $\gamma_{r2}(G) \leq \gamma_{r2}(G') + 2$ .

Let  $g$  be a Roman dominating function of  $G'$ . If  $g(x) = 2$  and  $g(y) = 0$ , then setting  $g(u) = g(v) = 0$  and  $g(w) = 2$  extends  $g$  to a Roman dominating function of  $G$ . Now we assume that  $\{g(x), g(y)\} \neq \{0, 2\}$ . Setting  $g(u) = g(w) = 0$  and  $g(v) = 2$  extends  $g$  to a Roman dominating function of  $G$ . By symmetry, this implies  $\gamma_R(G) \leq \gamma_R(G') + 2$ .  $\square$

A *spider* is a graph that arises by iteratively subdividing the edges of a star of order at least 4 arbitrarily often. The *center* of a spider is its unique vertex of degree at least 3. The *legs* of a spider are the maximal paths starting at its center. A leg is *good* if its length is not a multiple of 3.

**Lemma 7** *If  $G$  is a spider with at least three good legs, then  $\gamma_{r2}(G) + \gamma_R(G) \leq \frac{4}{3}n(G)$ .*

*Proof:* If some leg of  $G$  has length at least 3, then contracting three edges of this leg yields a spider  $G'$  with at least three good legs such that  $n(G') = n(G) - 3$ , and  $\gamma_{r2}(G) + \gamma_R(G) \leq \gamma_{r2}(G') + \gamma_R(G') + 4$ . By an inductive argument, we may therefore assume that all legs of  $G$  have length 1 or 2. Let  $G$  have exactly  $\ell_1$  legs of length 1 and exactly  $\ell_2$  legs of length 2. By the hypothesis,  $\ell_1 + \ell_2 \geq 3$ .

If  $\ell_1 = 0$ , then  $\gamma_{r2}(G) + \gamma_R(G) \leq (1 + \ell_2) + (2 + \ell_2) = 3 + 2\ell_2$  and  $n(G) = 1 + 2\ell_2$ . Since  $\ell_2 \geq 3$ , we obtain  $\frac{3+2\ell_2}{1+2\ell_2} \leq \frac{9}{7} < \frac{4}{3}$ , and hence  $\gamma_{r2}(G) + \gamma_R(G) \leq \frac{4}{3}n(G)$ . Now let  $\ell_1 \geq 1$ . We have  $\gamma_{r2}(G) + \gamma_R(G) \leq (2 + \ell_2) + (2 + \ell_2) = 4 + 2\ell_2$  and  $n(G) = 1 + \ell_1 + 2\ell_2$ . Since  $\frac{4+2\ell_2}{1+\ell_1+2\ell_2} < \frac{4+2\ell_2}{1+(\ell_1-1)+2\ell_2}$ , we may assume that either  $\ell_1 = 1$  or  $\ell_1 \geq 2$  and  $\ell_1 + \ell_2 = 3$ .

If  $\ell_1 = 1$ , then  $\ell_2 \geq 2$ , and we obtain  $\frac{4+2\ell_2}{1+\ell_1+2\ell_2} = \frac{4+2\ell_2}{2+2\ell_2} \leq \frac{4}{3}$ . If  $\ell_1 = 2$ , then  $\ell_2 = 1$ , and we obtain  $\frac{4+2\ell_2}{1+\ell_1+2\ell_2} = \frac{4}{5}$ . If  $\ell_1 = 3$ , then  $\ell_2 = 0$ , and we obtain  $\frac{4+2\ell_2}{1+\ell_1+2\ell_2} = \frac{4}{4}$ . Therefore, in all these cases,  $\gamma_{r2}(G) + \gamma_R(G) \leq \frac{4}{3}n(G)$ .  $\square$

We proceed to the proof of our first result.

*Proof of Theorem 3:* Let  $G$  be a counterexample of minimum order. A *branch vertex* is a vertex of degree at least 3. A *thread* is either a path between two branch vertices whose internal vertices have degree 2 or a cycle with exactly one branch vertex. By Lemma 6, every thread that is a path has length at most 4, and every thread that is a cycle has length at most 5.

Since  $\gamma_{r2}(C_3) + \gamma_R(C_3) = 2 + 2 \leq \frac{4}{3} \cdot 3$ ,  $\gamma_{r2}(C_4) + \gamma_R(C_4) = 2 + 3 \leq \frac{4}{3} \cdot 4$ ,  $\gamma_{r2}(C_8) + \gamma_R(C_8) = 4 + 6 \leq \frac{4}{3} \cdot 8$ , and, by Lemma 6,  $\gamma_{r2}(C_n) + \gamma_R(C_n) \leq \gamma_{r2}(C_{n-3}) + \gamma_R(C_{n-3}) + 4$  for  $n \geq 6$ , a simple inductive argument implies  $\gamma_{r2}(C_n) + \gamma_R(C_n) \leq \frac{4}{3}n$  for every  $n \geq 3$  that is distinct from 5. Hence, the maximum degree of  $G$  is at least 3.

If  $G$  arises from  $C_5$  by adding at least one edge, then  $\gamma_{r2}(G) + \gamma_R(G) \leq 3 + 3 < \frac{4}{3}n(G)$ . Hence, we may assume that  $G$  does not have this structure. Since removing edges can not decrease any

of the two domination parameters, we may assume that every edge between two branch vertices is a bridge, that is, an edge whose removal increases the number of components. Therefore, again by Lemma 6, no thread that is a path has length 4, that is, every thread that is a path has length 1, 2, or 3.

If  $u$  and  $v$  are adjacent branch vertices and none of the two components  $G_1$  and  $G_2$  of  $G - uv$  is  $C_5$ , then the choice of  $G$  implies  $\gamma_{r2}(G) + \gamma_R(G) \leq (\gamma_{r2}(G_1) + \gamma_R(G_1)) + (\gamma_{r2}(G_2) + \gamma_R(G_2)) \leq \frac{4}{3}n(G_1) + \frac{4}{3}n(G_2) = \frac{4}{3}n(G)$ . Hence, for every edge  $uv$  between branch vertices  $u$  and  $v$ ,  $G - uv$  contains a component that is  $C_5$ .

**Claim 1** *No two branch vertices are joined by two threads of length 2.*

*Proof of Claim 1:* For a contradiction, we assume that the two branch vertices  $x$  and  $y$  are joined by two threads of length 2, that is,  $x$  and  $y$  have at least two common neighbors of degree 2. Note that  $x$  and  $y$  are not adjacent and do not have a common neighbor of degree at least 3. Let  $G'$  arise from  $G$  by contracting all edges incident with the common neighbors of  $x$  and  $y$  to form a new vertex  $z$ . Since no thread of  $G$  is a path of length 5,  $G'$  is not  $C_5$ .

First we assume that  $N_G(x) \setminus N_G(y), N_G(y) \setminus N_G(x) \neq \emptyset$ . In this case,  $\delta(G') \geq 2$  and  $n(G') \leq n(G) - 3$ . Let  $f$  be a 2-rainbow dominating function of  $G'$ . If  $f(z) = \emptyset$ , then setting  $f(x) = \{1\}$ ,  $f(y) = \{2\}$ , and  $f(u) = \emptyset$  for  $u \in N_G(x) \cap N_G(y)$  extends  $f$  to a 2-rainbow dominating function of  $G$ . If  $f(z) \neq \emptyset$ , then setting  $f(x) = f(z)$ ,  $f(y) = \{1, 2\}$ , and  $f(u) = \emptyset$  for  $u \in N_G(x) \cap N_G(y)$  extends  $f$  to a 2-rainbow dominating function of  $G$ . In both cases, we obtain  $\gamma_{r2}(G) \leq \gamma_{r2}(G') + 2$ . Similarly, we obtain  $\gamma_R(G) \leq \gamma_R(G') + 2$ . Therefore, by the choice of  $G$ ,  $\gamma_{r2}(G) + \gamma_R(G) \leq \gamma_{r2}(G') + \gamma_R(G') + 4 \leq \frac{4}{3}n(G') + 4 \leq \frac{4}{3}n(G)$ .

Next we assume that  $N_G(x) \setminus N_G(y) = \emptyset$ . This implies that  $x$  and  $y$  have at least three common neighbors. Let  $G''$  arise from  $G'$  by adding two new vertices  $z'$  and  $z''$ , and adding the three new edges  $zz'$ ,  $z'z''$ , and  $z''z$ . Clearly,  $\delta(G'') \geq 2$ ,  $n(G'') \leq n(G) - 2$ , and  $G''$  is not  $C_5$ . Let  $f$  be a 2-rainbow dominating function of  $G''$ . Setting  $f(x) = \{1\}$ ,  $f(y) = \{1, 2\}$ , and  $f(u) = \emptyset$  for  $u \in N_G(x) \cap N_G(y)$  extends  $f$  to a 2-rainbow dominating function of  $G$ . Since  $|f(z)| + |f(z')| + |f(z'')| \geq 2$ , this implies  $\gamma_{r2}(G) \leq \gamma_{r2}(G'') + 1$ . Similarly, we obtain  $\gamma_R(G) \leq \gamma_R(G'') + 1$ . Therefore, by the choice of  $G$ ,  $\gamma_{r2}(G) + \gamma_R(G) \leq \gamma_{r2}(G'') + \gamma_R(G'') + 2 \leq \frac{4}{3}n(G'') + 2 < \frac{4}{3}n(G)$ .  $\square$

**Claim 2** *No two branch vertices are joined by two threads of length 3.*

*Proof of Claim 2:* For a contradiction, we assume that the two branch vertices  $x$  and  $y$  are joined by two threads of length 3. Note that  $x$  and  $y$  are not adjacent and do not have a common neighbor of degree at least 3, that is, every common neighbor of  $x$  and  $y$  is the internal vertex of a thread of length 2 between  $x$  and  $y$ . Let  $G'$  arise from  $G$  by contracting the edges of all threads between  $x$  and  $y$  to form a new vertex  $z$ , adding two new vertices  $z'$  and  $z''$ , and adding the three new edges  $zz'$ ,  $z'z''$ , and  $z''z$ . Clearly,  $\delta(G') \geq 2$ ,  $n(G') \leq n(G) - 3$ , and  $G'$  is not  $C_5$ .

Let  $f$  be a 2-rainbow dominating function of  $G'$ . Setting  $f(x) = f(y) = \{1, 2\}$  and  $f(u) = \emptyset$  for  $u \in V(C) \setminus \{x, y\}$  extends  $f$  to a 2-rainbow dominating function of  $G$ . Since  $|f(z)| + |f(z')| + |f(z'')| \geq 2$ , this implies  $\gamma_{r2}(G) \leq \gamma_{r2}(G') + 2$ . Similarly, we obtain  $\gamma_R(G) \leq \gamma_R(G') + 2$ . Therefore, by the choice of  $G$ ,  $\gamma_{r2}(G) + \gamma_R(G) \leq \gamma_{r2}(G') + \gamma_R(G') + 4 \leq \frac{4}{3}n(G') + 4 \leq \frac{4}{3}n(G)$ .  $\square$

**Claim 3**  *$G$  does not have a thread  $C$  that is a cycle of length 5 whose branch vertex  $u$  has degree exactly 3 and is joined by a thread  $P$  of length 2 or 3 to another branch vertex  $v$ .*

*Proof of Claim 3:* Let  $G' = G[(V(C) \cup V(P)) \setminus \{v\}]$  and  $G'' = G - V(G')$ . Since  $G'$  contains a spanning subgraph that is a spider with three good legs, Lemma 7 implies  $\gamma_{r2}(G') + \gamma_R(G') \leq \frac{4}{3}|V(G')|$ . If  $G''$  is not  $C_5$ , then, by the choice of  $G$ ,  $\gamma_{r2}(G) + \gamma_R(G) \leq (\gamma_{r2}(G') + \gamma_R(G')) + (\gamma_{r2}(G'') + \gamma_R(G'')) \leq \frac{4}{3}|V(G')| + \frac{4}{3}(n(G) - |V(G')|) = \frac{4}{3}n(G)$ . If  $G''$  is  $C_5$  and  $P$  has length 2, then  $\gamma_{r2}(G) + \gamma_R(G) = 6 + 8 < \frac{4}{3} \cdot 11 = \frac{4}{3}n(G)$ . Finally, if  $G''$  is  $C_5$  and  $P$  has length 3, then  $\gamma_{r2}(G) + \gamma_R(G) = 8 + 8 \leq \frac{4}{3} \cdot 12 = \frac{4}{3}n(G)$ .  $\square$

**Claim 4** *There is a set  $E$  of edges such that  $G - E$  has one component  $S$  that is a spider with only good legs and all remaining components of  $G - E$  have minimum degree at least 2 and are distinct from  $C_5$ .*

*Proof of Claim 4:* If  $G$  consists of two copies of  $C_5$  together with one bridge, then  $\gamma_{r2}(G) + \gamma_R(G) = 6 + 7 < \frac{4}{3} \cdot 10 = \frac{4}{3}n(G)$ . Hence,  $G$  does not have this structure. Together with Claim 3, this implies that  $G$  has some branch vertex  $c$  such that either  $c$  does not lie on a thread that is a cycle of length 5 or  $c$  has degree at least 4. We will describe the construction of  $E$  starting with the empty set such that  $c$  is the center of the spider  $S$  and  $S$  has  $d_G(c)$  good legs, that is, every edge of  $G$  incident with  $c$  will be the initial edge of a leg of  $S$ .

- For every thread  $C$  of  $G$  that is a cycle and contains  $c$ , add to  $E$  exactly one edge of  $C$  at maximum distance from  $c$ . Since  $C$  has length 3, 4, or 5, this leads to two good legs for  $S$ .
- For every thread  $P$  of  $G$  that is a path of length 1 and contains  $c$ , the only edge  $e$  of  $P$  is a bridge and the component  $K$  of  $G - e$  that contains the neighbor of  $c$  is  $C_5$ . Add to  $E$  exactly one edge of  $K$  that is incident with a neighbor of  $c$ . This leads to one good leg of length 5 for  $S$ .

Let  $U$  be the set of branch vertices of  $G$  that are joined to  $c$  by at least one thread that is a path of length at least 2. Note that  $U$  contains no neighbor of  $c$ . By Claims 1 and 2, no vertex in  $U$  is joined to  $c$  by more than two threads, and if some vertex  $U$  is joined to  $c$  by two threads, then these two threads have length 2 and 3, respectively. For  $u$  in  $U$ , let  $E(u)$  denote the edges incident with  $u$  that lie on threads between  $c$  and  $u$ . For a branch vertex  $u$  distinct from  $c$ , let  $p(u)$  be the number of threads between  $c$  and  $u$ . Note that for  $u \in U$ , we have  $p(u) = |E(u)|$ .

- Let  $u_1$  be a vertex of degree exactly 3 with  $p(u_1) = 2$ .

Let  $Q_1$  denote the unique third thread starting at  $u_1$  and joining  $u_1$  with some vertex  $u_2$  distinct from  $c$ . Let  $u_1, u_2, \dots, u_k$  be a maximal sequence of distinct branch vertices such that for  $2 \leq i \leq k-1$ , the vertex  $u_i$  has degree exactly  $p(u_i) + 2$  and  $u_i$  is joined to  $u_{i+1}$  by exactly one thread  $Q_i$ . Since the sequence  $u_1, u_2$  with only two elements satisfies all requirements trivially, such a maximal sequence is well defined. We call such a maximal sequence *special*.

Note that for  $2 \leq i \leq k-1$ , the vertex  $u_i$  belongs to  $U$ , and is joined to  $u_{i-1}$  by the thread  $Q_{i-1}$  and to  $u_{i+1}$  by the thread  $Q_i$ . Furthermore, note that  $u_k$  not necessarily belongs to  $U$ .

By the maximality of the sequence, the vertex  $u_k$  has either degree  $p(u_k) + 1$  or degree at least  $p(u_k) + 3$ .

- First we assume that  $d_G(u_k) = p(u_k) + 1$ .

Add to  $E$  the set  $E(u_i)$  for every  $2 \leq i \leq k$ . Since the two threads between  $c$  and  $u_1$  have length 2 and 3, adding to  $E$  one of the two edges in  $E(u_1)$  leads to  $p(u_1) + \dots + p(u_k)$  good legs for  $S$ .

- Next we assume that  $d_G(u_k) \geq p(u_k) + 3$ .

If  $u_k$  has degree exactly  $p(u_k) + 3$  and belongs to a thread  $C$  that is a cycle of length 5, then add to  $E$

- \*  $E(u_i)$  for every  $2 \leq i \leq k$ ,
- \* one edge of  $C$  incident with  $u_k$ , and
- \* one of the two edges in  $E(u_1)$ .

Again the edge from  $E(u_1)$  can be chosen such that we obtain  $p(u_1) + \dots + p(u_k)$  good legs for  $S$ .

If  $u_k$  does not have degree exactly  $p(u_k) + 3$  or does not belong to a thread  $C$  that is a cycle of length 5, then add to  $E$

- \*  $E(u_i)$  for every  $2 \leq i \leq k$ ,
- \* the edge of  $Q_{k-1}$  incident with  $u_k$ , and
- \* one of the two edges in  $E(u_1)$ .

Again the edge from  $E(u_1)$  can be chosen such that we obtain  $p(u_1) + \dots + p(u_k)$  good legs for  $S$ .

It remains to consider the vertices in  $U$  that do not lie in some special sequence. Note that each such vertex  $u$  satisfies either  $d_G(u) \geq 4$  or  $d_G(u) = 3$  and  $p(u) = 1$ .

- Let  $u$  in  $U$  be such that  $d_G(u) \geq 4$ ,  $p(u) = 1$ , and  $u$  does not belong to some special sequence.

Add to  $E$  the unique edge in  $E(u)$ . This leads to one good leg for  $S$ .

- Let  $u$  in  $U$  be such that  $d_G(u) \geq 4$ ,  $p(u) = 2$ , and  $u$  does not belong to some special sequence.

If  $d_G(u) = 4$  and  $u$  belongs to a thread  $C$  that is a cycle of length 5, then add to  $E$  one edge of  $C$  incident with  $u$  and one of the two edges in  $E(u)$ . Again, this last edge can be chosen such that we obtain two good legs for  $S$ .

If  $d_G(u) > 4$  or  $u$  does not belong to a thread  $C$  that is a cycle of length 5, then add to  $E$  the set  $E(u)$ . This leads to two good legs for  $S$ .

- Finally, let  $u$  in  $U$  be such that  $d_G(u) = 3$ ,  $p(u) = 1$ , and  $u$  does not belong to some special sequence.

By Claim 3,  $u$  does not belong to a thread  $C$  that is a cycle of length 5. Add to  $E$  the unique edge in  $E(u)$ . This leads to one good leg for  $S$ .

This completes the construction of  $E$ . As argued above, the choice of  $c$  implies that we obtain one good leg for  $S$  for every edge of  $G$  incident with  $c$ . Furthermore, the construction of  $E$  easily implies that  $G - V(S)$  has the desired properties.  $\square$

Let  $S$  be in Claim 4 and let  $R = G - V(S)$ . By the choice of  $G$  and Lemma 7,  $\gamma_{r2}(G) + \gamma_R(G) \leq (\gamma_{r2}(S) + \gamma_R(S)) + (\gamma_{r2}(R) + \gamma_R(R)) \leq \frac{4}{3}n(S) + \frac{4}{3}n(R) = \frac{4}{3}n(G)$ , which completes the proof.  $\square$

We proceed to our second result, the characterization of all extremal graphs for Theorem 2. First we characterize the extremal trees.

For  $k \in \mathbb{N}$ , let  $\mathcal{T}_k$  be the set of all trees  $T$  that arise from  $k$  disjoint copies  $a_1b_1c_1d_1, \dots, a_kb_kc_kd_k$  of the path of order 4 by adding some edges between vertices in  $\{b_1, \dots, b_k\}$ , that is,  $T[\{b_1, \dots, b_k\}]$  is a tree. Let  $\mathcal{T} = \bigcup_{k \in \mathbb{N}} \mathcal{T}_k$ .

**Lemma 8** *For  $k \in \mathbb{N}$ ,  $\gamma_{r2}(T) = \gamma_R(T) = 3k$  for every  $T \in \mathcal{T}_k$ .*

*Proof:* Let  $T \in \mathcal{T}_k$ . We denote the vertices of  $T$  as in the definition of  $\mathcal{T}_k$ .

Since

$$f_k : V(T) \rightarrow 2^{\{1,2\}} : u \mapsto \begin{cases} \emptyset & , \text{ if } u \in \{a_1, \dots, a_k\} \cup \{c_1, \dots, c_k\}, \\ \{1, 2\} & , \text{ if } u \in \{b_1, \dots, b_k\}, \text{ and} \\ \{1\} & , \text{ if } u \in \{d_1, \dots, d_k\} \end{cases}$$

and

$$g_k : V(T) \rightarrow \{0, 1, 2\} : u \mapsto \begin{cases} 0 & , \text{ if } u \in \{a_1, \dots, a_k\} \cup \{c_1, \dots, c_k\}, \\ 2 & , \text{ if } u \in \{b_1, \dots, b_k\}, \text{ and} \\ 1 & , \text{ if } u \in \{d_1, \dots, d_k\} \end{cases}$$

are a 2-rainbow dominating function and a Roman dominating function of  $T$ , respectively, we obtain  $\gamma_{r2}(T), \gamma_R(T) \leq 3k$ .

For every 2-rainbow dominating function  $f$  and every Roman dominating function  $g$  of  $T$ , and for every  $1 \leq i \leq k$ , it is easy to see that  $|f(a_i)| + |f(b_i)| + |f(c_i)| + |f(d_i)| \geq 3$  and  $g(a_i) + g(b_i) + g(c_i) + g(d_i) \geq 3$ . Hence, we obtain  $\gamma_{r2}(T), \gamma_R(T) \geq 3k$ .  $\square$



The following result is a strengthened version of Theorem 2.8 in [4].

**Lemma 9** *If  $T$  is a tree of order  $n(T)$  at least 3, then  $\gamma_{r2}(T) + \gamma_R(T) \leq \frac{6}{4}n(T)$  with equality if and only if  $T \in \mathcal{T}$ .*

*Proof:* We prove the statement by induction on the order of  $T$ . If  $T$  is a path of order at most 5, then the statement is easily verified. If  $T$  is a spider with at least three good legs, then Lemma 7 implies  $\gamma_{r2}(T) + \gamma_R(T) \leq \frac{4}{3}n(T) < \frac{6}{4}n(T)$ . Hence, we may assume that  $T$  is not any of these trees.

Let  $u_1u_2 \dots u_\ell$  be a longest path in  $T$  such that  $d_G(u_2)$  is maximum possible. Clearly,  $\ell \geq 4$ .

If  $d_T(u_2) \geq 3$ , then let  $T' = T - (N_G[u_2] \setminus \{u_3\})$ . Clearly,  $\gamma_{r2}(T) \leq \gamma_{r2}(T') + 2$  and  $\gamma_R(T) \leq \gamma_R(T') + 2$ . Since  $T$  is not a spider with at least three good legs,  $T'$  has order at least 3. Hence, by induction,  $\gamma_{r2}(T') + \gamma_R(T') \leq 4 + \gamma_{r2}(T') + \gamma_R(T') \leq 4 + \frac{6}{4}n(T') \leq 4 + \frac{6}{4}(n(T) - 3) < \frac{6}{4}n(T)$ . Hence, we may assume that  $d_T(u_2) = 2$ .

Let  $T_1$  and  $T_2$  be the components of  $T - u_3u_4$  such that  $T_1$  contains  $u_3$ . Clearly,  $T_1$  is either a path of order between 3 and 5, or a spider with at least three good legs. Since  $T$  is not a path of order at most 5 or a spider with at least three good legs,  $T_2$  has order at least 3. If  $T_1$  is not a path of order 4, then  $T \notin \mathcal{T}$  and, by induction,  $\gamma_{r2}(T) + \gamma_R(T) \leq (\gamma_{r2}(T_1) + \gamma_R(T_1)) + (\gamma_{r2}(T_2) + \gamma_R(T_2)) < \frac{6}{4}n(T_1) + \frac{6}{4}n(T_2) = \frac{6}{4}n(T)$ . Hence, we may assume that  $T_1$  is a path of order 4. By a similar argument we obtain  $T_2 \in \mathcal{T}$ . Let  $T_2 \in \mathcal{T}_{k-1}$  and denote the vertices of  $T_2$  as in the definition of  $\mathcal{T}_{k-1}$ . Let  $T_1$  be  $a_kb_kc_kd_k$  such that  $u_3 = b_k$ . Let  $a_ib_ic_id_i$  be the subpath of order 4 of  $T_2$  that contains  $u_4$ .

If either  $u_4 = b_i$  or  $k = 2$  and  $u_4 = c_i$ , then  $T \in \mathcal{T}$  and Lemma 8 implies the desired result. Hence, we may assume that these cases do not occur.

Let  $g_k$  be the function from the proof of Lemma 8. Clearly,  $g_k$  is a Roman dominating function of  $T$ . We will modifying  $g_k$  on the four vertices in  $\{a_i, b_i, c_i, d_i\}$  in such a way that we obtain a Roman dominating function  $g$  of  $T$  of smaller weight than  $g_k$ . Once this is done, we obtain  $\gamma_{r2}(T) \leq \gamma_R(T) < 3k = \frac{3}{4}n(T)$  and hence  $\gamma_{r2}(T) + \gamma_R(T) < \frac{6}{4}n(T)$ .

If  $u_4 = a_i$ , then we change  $g_k$  to  $g$  such that  $(g(a_i), g(b_i), g(c_i), g(d_i)) = (0, 0, 2, 0)$ . If  $u_4 = c_i$ , then we may assume that  $k \geq 3$ , and we change  $g_k$  to  $g$  such that  $(g(a_i), g(b_i), g(c_i), g(d_i)) = (1, 0, 0, 1)$ . Finally, if  $u_4 = d_i$ , then we change  $g_k$  to  $g$  such that  $(g(a_i), g(b_i), g(c_i), g(d_i)) = (0, 2, 0, 0)$ . It is straightforward to check that  $g$  has the desired properties, which completes the proof.  $\square$

Let the class  $\mathcal{G}$  of connected graphs be such that a connected graph  $G$  belongs to  $\mathcal{G}$  if and only if  $G$  arises

- either from the unique tree in  $\mathcal{T}_2$  by adding the edge  $c_1c_2$
- or from some tree in  $\mathcal{T}_k$  by arbitrarily adding edges between vertices in  $\{b_1, \dots, b_k\}$ ,

where we denote the vertices as in the definition of  $\mathcal{T}_k$ .

**Theorem 10** *If  $G$  is a connected graph of order  $n(G)$  at least 3, then  $\gamma_{r2}(G) + \gamma_R(G) \leq \frac{6}{4}n(G)$  with equality if and only if  $G \in \mathcal{G}$ .*

*Proof:* Similarly as in the proof of Lemma 8, we obtain  $\gamma_{r2}(G) + \gamma_R(G) = \frac{6}{4}n(G)$  for every graph  $G \in \mathcal{G}$ . Now let  $G$  be a connected graph of order at least 3. Let  $T$  be a spanning tree of  $G$ . By Lemma 9,  $\gamma_{r2}(G) + \gamma_R(G) \leq \gamma_{r2}(T) + \gamma_R(T) \leq \frac{6}{4}n(G)$ . Now, we assume that  $\gamma_{r2}(G) + \gamma_R(G) = \frac{6}{4}n(G)$ . This implies  $\gamma_{r2}(T) + \gamma_R(T) = \frac{6}{4}n(T)$ . By Lemma 9, we obtain  $T \in \mathcal{T}_k$  for some  $k \in \mathbb{N}$ . We denote the vertices of  $T$  as in the definition of  $\mathcal{T}$ . Note that  $T[\{b_1, \dots, b_k\}]$  is connected. Let  $f_k$  be the function from the proof of Lemma 8. Let  $G_{\leq j}$  be the subgraph of  $G$  induced by  $\bigcup_{i=1}^j \{a_i, b_i, c_i, d_i\}$ .

First, we assume that  $a_1$  has degree at least 2 in  $G$ . If  $a_1$  is adjacent to  $c_1$  or  $d_1$ , then we have  $\gamma_{r2}(G_{\leq 1}) = 2$ , and replacing the values of  $f_k$  on the vertices of  $G_{\leq 1}$  by the values of an optimal 2-rainbow dominating function of  $G_{\leq 1}$  implies  $\gamma_{r2}(G) < 3k$ , and hence  $\gamma_{r2}(G) + \gamma_R(G) \leq \gamma_{r2}(T) + \gamma_R(T) < 6k = \frac{6}{4}n(G)$ , which is a contradiction. If  $a_1$  is adjacent some vertex in  $\{a_2, b_2, c_2, d_2\}$ , then we have  $\gamma_{r2}(G_{\leq 2}) \leq 5$ , and replacing the values of  $f_k$  on the vertices of  $G_{\leq 2}$  by the values of an optimal 2-rainbow dominating function of  $G_{\leq 2}$  implies  $\gamma_{r2}(G) < 3k$ , which implies the same contradiction as above. Hence, by symmetry, we may assume that all vertices in  $\{a_1, \dots, a_k\} \cup \{d_1, \dots, d_k\}$  have degree 1 in  $G$ .

If  $b_1$  is adjacent to  $c_2$ , then we change the values of  $f_k$  on the vertices in  $\{a_2, b_2, c_2, d_2\}$  to obtain a function  $f$  with  $(f(a_2), f(b_2), f(c_2), f(d_2)) = (\{1\}, \emptyset, \emptyset, \{1\})$ . Note that  $b_2$  is adjacent to some vertex  $b_i$  for  $i \neq 2$  and that  $f_k(b_i) = \{1, 2\}$ . Hence  $f$  is a 2-rainbow dominating function of  $G$  and we obtain  $\gamma_{r2}(G) < 3k$ , which implies the same contradiction as above. Hence, by symmetry, we may assume that no vertex  $b_i$  is adjacent to some vertex  $c_j$  for distinct  $i$  and  $j$ .

If  $b_1$  is adjacent to  $b_2$  and  $c_2$  is adjacent to  $c_3$ , then

$$f : V(G_{\leq 3}) \rightarrow 2^{\{1,2\}} : u \mapsto \begin{cases} \emptyset & , \text{ if } u \in \{a_1, c_1, b_2, c_2, b_3, d_3\}, \\ \{1\} & , \text{ if } u \in \{d_1, a_2, d_2, a_3\}, \text{ and} \\ \{1, 2\} & , \text{ if } u \in \{b_1, c_3\} \end{cases}$$

is a 2-rainbow dominating function of  $G_{\leq 3}$  of weight 8. This implies  $\gamma_{r2}(G) < 3k$ , which implies the same contradiction as above. Hence, by symmetry, we may assume that if some vertex  $c_i$  has degree more than 2, then  $b_i$  and  $c_i$  both have degree exactly 3 and there is some index  $j$  distinct from  $i$  such that  $b_i$  is adjacent to  $b_j$  and  $c_i$  is adjacent to  $c_j$ .

Altogether, the above observations easily imply that  $G \in \mathcal{G}$  as desired.  $\square$

**Acknowledgment** J.D. Alvarado and S. Dantas were partially supported by FAPERJ, CNPq, and CAPES.

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